JOURNAL OF COMPUTATIONAL PHYSICS 39, 433–442 (1981)

Variable Mesh Cubic Spline Technique for N-Wave Solution of Burgers' Equation

B. L. LOHAR AND P. C. JAIN

Department of Mathematics, Indian Institute of Technology, Powai, Bombay 400076, India

Received September 18, 1979; revised March 20, 1980

A variable mesh cubic spline technique is developed for the shock-like solution of the one-dimensional Burgers' equation so that the necessity of taking very fine mesh all over the computational region could be avoided. The scheme is employed, by taking a typical initial solution, to study the propagation of plane N-waves. Results are presented for various values of the dissipative parameter δ. Numerical results are in good agreement with the known exact results.

1. INTRODUCTION

The equation under discussion in this paper is the following:

$$u_t + uu_x = \frac{\delta}{2} u_{xx}, \tag{1}$$

where $u = u(x, t)$ and δ is the coefficient of diffusivity, also known as the dissipative parameter. Burgers [1] proposed (1) as model for one-dimensional turbulence and shock waves, and subsequently it is called Burgers' equation. This equation admits the transformation

$$u = -\delta \left(\frac{\Phi_x}{\Phi} \right)$$

known as the Hopf–Cole [2, 3] transformation and gives the well-known heat equation in Φ. Cole [3] described in details the general properties of (1). Lighthill [4] derived (1) from the basic equation in gas dynamics and gave the analytical solution, the so-called N-wave solution, where the effect of diffusion is confined to two thin boundary layers each of thickness $(\delta t)^{1/2}$, corresponding to head and tail shocks in explosions or sonic boom theory.

When the flow characteristics are continuous, the uniform mesh finite difference methods can be used for solving problems over a regular domain in fluid mechanics and allied areas. But, in the case that the flow characteristics have discontinuities, the uniform mesh techniques may not provide realistic results unless a very fine mesh is

0021-9991/81/020433–10$02.00/0

used. Normally, a very fine mesh is required in the neighborhoods of the points of discontinuity. Sachdev and Seebass [5] used an implicit predictor-corrector finite difference method of Douglas and Jones [6] for the solution of (1) to study the propagation of plane N-waves. They [5] used the uniform mesh of size 10^{-2} in both x and t directions and gave the results for $\delta = 10^{-2}$. They showed that the uniform mesh of size 10^{-2}, which is of $O(\delta)$, is sufficient for the numerical results. Nevertheless, in practice we come across even smaller values of δ and in that case even if we take the uniform mesh of size of $O(\delta)$ it may not be possible to solve the problem.

Jain and Holla [7] and Jain and Lohar [8] described the cubic spline techniques for solving (1) using uniform mesh. These techniques also could not be used for the shock-like solution of (1) since very fine mesh is needed in the neighborhood of the center of the shock. Recently, Chong [9] suggested a variable mesh finite difference method where he used the predictor–corrector formula of Douglas and Jones [6]. He described the concept of variable mesh and observed that inside the boundary layer or in the neighborhood of the points of discontinuity we need to use fine mesh, while outside the boundary layers, we can afford to use coarser mesh. Moreover, the change in the mesh size, i.e., from finer to coarser and vice versa, should be gradual.

In this paper, a new numerical method *Variable Mesh Cubic Spline Technique* (VMCST)is proposed for the plane N-wave solution of (1). Numerical solution obtained by the scheme is compared with the exact solution given by Lighthill [4] and is found to be in good agreement.

2. The Cubic Spline Technique

Consider the equation

$$u_t + f_x = \frac{\partial}{2} u_{xx}, \tag{2}$$

where $f = u^2/2$ is the homogeneous function of u of degree 2. We wish to find N-wave solution of (2) in the domain $[0 \leqslant x < c] \times [0 \leqslant t \leqslant T]$, where c is a positive quantity such that $u(x, t) \leqslant 10^{-8}$ for $x \geqslant c$ and T is a fixed value of time t.

It is known that a complicated problem can be solved more efficiently by using the splitting technique. Equation (2) is a nonlinear parabolic equation. We split (2) into two equations as follows [8]:

$$\frac{1}{2} u_t = -f_x, \tag{3}$$

$$\frac{1}{2} u_t = \frac{\delta}{2} u_{xx}. \tag{4}$$

Approximating the time derivative by forward difference and the space derivative by the first-order derivatives of the cubic spline function, (3) is approximated as

$$\theta_1 m_i^{n+1/2} + (1-\theta_1)\, m_i^n = -\frac{1}{k}\,(U_i^{n+1/2} - U_i^n), \tag{5}$$

where $\theta_1 \in [0, 1]$ is cubic spline parameter, $m_i^n = S_n'(x_i)$, $S_n(x)$ is the cubic spline function, U_i^n is the discrete approximation of $u(x, t)$ at (x_i, t_n), $i = 0, 1, \ldots, I$; $n = 0, 1, \ldots,$ and k is the time step.

We take the cubic spline relations [10]

$$\begin{aligned} &\frac{1}{h_i} m_{i-1}^n + 2\left(\frac{1}{h_i} + \frac{1}{h_{i+1}}\right) m_i^n + \frac{1}{h_{i+1}} m_{i+1}^n \\ &\quad = \frac{3}{h_i^2}(f_i^n - f_{i-1}^n) + \frac{3}{h_{i+1}^2}(f_{i+1}^n - f_i^n), \end{aligned} \tag{6}$$

$$\begin{aligned} &\frac{1}{h_i} m_{i-1}^{n+1/2} + 2\left(\frac{1}{h_i} + \frac{1}{h_{i+1}}\right) m_i^{n+1/2} + \frac{1}{h_{i+1}} m_{i+1}^{n+1/2} \\ &\quad = \frac{3}{h_i^2}(f_i^{n+1/2} - f_{i-1}^{n+1/2}) + \frac{3}{h_{i+1}^2}(f_{i+1}^{n+1/2} - f_i^{n+1/2}). \end{aligned} \tag{7}$$

Multiplying (5) by the factors $1/h_i$, $2[(1/h_i) + (1/h_{i+1})]$, and $1/h_{i+1}$, respectively, adding the resulting equations, and using (6) and (7), we get

$$\begin{aligned} &3\theta_1 \left\{\frac{1}{h_i^2}(f_i^{n+1/2} - f_{i-1}^{n+1/2}) + \frac{1}{h_{i+1}^2}(f_{i+1}^{n+1/2} - f_i^{n+1/2})\right\} \\ &\quad + 3(1-\theta_1)\left\{\frac{1}{h_i^2}(f_i^n - f_{i-1}^n) + \frac{1}{h_{i+1}^2}(f_{i+1}^n - f_i^n)\right\} \\ &\quad = -\frac{1}{k}\left\{\frac{1}{h_i}(U_{i-1}^{n+1/2} - U_{i-1}^n) + 2\left(\frac{1}{h_i} + \frac{1}{h_{i+1}}\right)(U_i^{n+1/2} - U_i^n) \right. \\ &\quad\quad \left. + \frac{1}{h_{i+1}}(U_{i+1}^{n+1/2} - U_{i+1}^n)\right\}. \end{aligned} \tag{8}$$

Transposing all the terms having superscript $n + \frac{1}{2}$ to the left-hand side and simplifying, we get

$$\begin{aligned} &\{h_{i+1} U_{i-1}^{n+1/2} + 2(h_i + h_{i+1})\, U_i^{n+1/2} + h_i U_{i+1}^{n+1/2}\} \\ &\quad + \frac{3\theta_1 k}{h_i h_{i+1}} \{h_{i+1}^2 f_{i-1}^{n+1/2} + (h_{i+1}^2 - h_i^2) f_i^{n+1/2} - h_i^2 f_{i+1}^{n+1/2}\} \\ &\quad = \{h_{i+1} U_{i-1}^n + 2(h_i + h_{i+1})\, U_i^n + h_i U_{i+1}^n\} \\ &\quad\quad - \frac{3(1-\theta_1)\, k}{h_i h_{i+1}} \{h_{i+1}^2 f_{i-1}^n + (h_{i+1}^2 - h_i^2) f_i^n - h_i^2 f_{i+1}^n\}. \end{aligned} \tag{9}$$

Again, approximating the time derivative by forward differences and the space derivative by second-order derivatives of cubic spline function, (4) is approximated as

$$\theta_2 M_i^{n+1} + (1-\theta_2) M_i^{n+1/2} = \frac{2}{\delta k}(U_i^{n+1} - U_i^{n+1/2}), \tag{10}$$

where $\theta_2 \in [0, 1]$ is the cubic spline parameter and $M_i^n = S_n''(x_i)$. We take the cubic spline relations [10]

$$\begin{aligned} &\tfrac{1}{6}h_i M_{i-1}^{n+1} + \tfrac{1}{3}(h_i + h_{i+1}) M_i^{n+1} + \tfrac{1}{6}h_{i+1} M_{i+1}^{n+1} \\ &\quad = \frac{1}{h_{i+1}}(U_{i+1}^{n+1} - U_i^{n+1}) - \frac{1}{h_i}(U_i^{n+1} - U_{i-1}^{n+1}), \end{aligned} \tag{11}$$

$$\begin{aligned} &\tfrac{1}{6}h_i M_{i-1}^{n+1/2} + \tfrac{1}{3}(h_i + h_{i+1}) M_i^{n+1/2} + \tfrac{1}{6}h_{i+1} M_{i+1}^{n+1/2} \\ &\quad = \frac{1}{h_{i+1}}(U_{i+1}^{n+1/2} - U_i^{n+1/2}) - \frac{1}{h_i}(U_i^{n+1/2} - U_{i-1}^{n+1/2}). \end{aligned} \tag{12}$$

Eliminating M_i^{n+1} and $M_i^{n+1/2}$ from (10) with the help of (11) and (12) in the similar manner as for the derivative of (9) from (5), (6), and (7), we get

$$\begin{aligned} &\left(h_i - \frac{3\delta k}{h_i}\theta_2\right) U_{i-1}^{n+1} + 2(h_i + h_{i+1})\left(1 + \frac{3}{2}\frac{\delta k}{h_i h_{i+1}}\theta_2\right) U_i^{n+1} \\ &\quad + \left(h_{i+1} - \frac{3\delta k}{h_{i+1}}\theta_2\right) U_{i+1}^{n+1} \\ &= \left\{h_i + \frac{3\delta k}{h_i}(1-\theta_2)\right\} U_{i-1}^{n+1/2} + 2(h_i + h_{i+1})\left\{1 - \frac{3}{2}\frac{\delta k}{h_i h_{i+1}}(1-\theta_2)\right\} U_i^{n+1/2} \\ &\quad + \left\{h_{i+1} + \frac{3\delta k}{h_{i+1}}(1-\theta_2)\right\} U_{i+1}^{n+1/2}. \end{aligned} \tag{13}$$

Equations (9) and (13) constitute an implicit two-step finite difference scheme for solving (2). This scheme leads to a system of nonlinear difference equations to be solved at each time step. One can remove the nonlinearity occurring in (9) and maintain the order of accuracy by the linearization procedure commonly used in the numerical solution of nonlinear systems of ordinary differential equations [11]. Equation (9) transforms to

$$\begin{aligned} &h_{i+1}\left(1 - \frac{3k}{h_i}\theta_1 A_{i-1}^n\right) U_{i-1}^{n+1/2} + (h_i + h_{i+1})\left\{2 + \frac{3k}{h_i h_{i+1}}\theta_1 (h_{i+1} - h_i) A_i^n\right\} \\ &\qquad \times U_i^{n+1/2} + h_i\left(1 + \frac{3k}{h_{i+1}}\theta_1 A_{i+1}^n\right) U_{i+1}^{n+1/2} \\ &\quad = \{h_{i+1} U_{i-1}^n + 2(h_i + h_{i+1}) U_i^n + h_i U_{i+1}^n\} \\ &\qquad - \frac{3k}{h_i h_{i+1}}(1-\theta_1)\{h_i^2 f_{i+1}^n + (h_{i+1}^2 - h_i^2) f_i^n - h_{i+1}^2 f_{i-1}^n\}, \end{aligned} \tag{14}$$

where

$$A = \frac{\partial f}{\partial u}. \tag{15}$$

Equations (13) and (14) yield tridiagonal systems which are solved by the Thomas algorithm [12].

3. N-Wave Solution of the Burgers' Equation

We consider the case of a balanced N-wave for which

$$\int_{-\infty}^{\infty} u\, dx = 0. \tag{16}$$

The Reynolds number of each lobe of the N-waves, known as the lobe Reynolds number is defined as [4],

$$R = \frac{1}{\delta} \int_{0}^{\infty} u\, dx. \tag{17}$$

The lobe Reynolds number at time t, for some t_0, is given by [4]

$$R = \log \left\{ 1 + \left(\frac{t_0}{t} \right)^{1/2} \right\}. \tag{18}$$

For the initial condition ($t = 1$),

$$u(x, 1) = \frac{x}{1 + t_0^{-1/2} \exp(x^2/2\delta)}, \tag{19}$$

the velocity profile at lobe Reynolds number R is

$$u(x, t) = \frac{x/t}{1 + \exp(x^2/2\delta t)/(e^R - 1)}. \tag{20}$$

By using (18)

$$u(x, t) = \frac{x/t}{1 + \exp(x^2/2\delta t)(t/t_0)^{1/2}}. \tag{21}$$

As the solution is antisymmetric in x and has two boundary layers of thickness $(\delta t)^{1/2}$ each corresponding to the head and tail shock of a sonic boom. We solve (1) for $x > 0$ only and take the left boundary condition as $u(0, t) = 0$, for all t. Since u decays exponentially for large x, for computational work we take a positive quantity

c such that $u(x, t) \leqslant 10^{-8}$ for $x \geqslant c$. As the solution progresses in t, the value of c changes so that the grid points are either increased or decreased (as the case may be) with $u = 0$.

4. Method of Solution

For generating the nonuniform mesh we briefly describe the procedure due to Chong [9]. To locate the center of the shock we take fine mesh of size $h/4$ or $h/5$ where $h = 10^{-2}$. The center of the shock occurs at the place where $|\Delta_x u|$ is maximum, say, at $x = x_M$ so that $|\Delta_x u_M|$ is maximum. The thickness of te shock, γ, is given by

$$\gamma = 1/|\Delta_x u_M|. \tag{22}$$

If $\{\bar{x}_i\}$ is the set of new grid points in the nonuniform mesh, which we shall generate, and $\bar{x}_{M'}$ is the center of the shock in the nonuniform mesh then $\bar{x}_{M'} = x_M$. We take

$$\bar{h}_{M'} = \alpha\gamma h; \tag{23}$$

where $\bar{h}_{M'} = \bar{x}_{M'} - \bar{x}_{M'-1}$ and α is a positive constant. This value of $\bar{h}_{M'}$ can be made much smaller than the value of h. We take L grid points of size $\bar{h}_{M'}$ on both sides of the center of the shock. Outside this region, i.e., $[\bar{x}_{M'} - L\bar{h}_{M'}, \bar{x}_{M'} + L\bar{h}_{M'}]$ we take the gradually increasing mesh defined by

$$\bar{h}_{M' \pm (L+i)} = (1 + \beta h)^i \, \bar{h}_{M'}, \qquad i = 1, 2, 3,..., \tag{24}$$

where $\beta > 0$. At the place where the mesh size becomes larger than h, we stop using (24) and take remaining mesh sizes to be h until $[0, c)$ is covered. For computational work we took $L = 50$ (it may be varied according to the shock thickness). In this procedure it may so happen that $x = 0$ is not a mesh point. In that case we take the first negative grid point $x = -b$ $(0 < b < h)$, and multiply all the mesh sizes by the factor $x_M/(x_M + b)$ so that $x = 0$ and $x = \bar{x}_{M'}$ are both the grid points on the new grid system.

Again, we find the center of the shock and the shock thickness in the new grid system. Let these values be $x_{M''}$ and γ' respectively. If $|M' - M''| > 15$ or $|\gamma - \gamma'| > 0.1$, we repeat the nonuniform mesh construction. For the computational work, the time step k_n at time $t = t_n$ is taken as $k_n = 10h_{M,n}$, where $h_{M,n}$ is the minimum mesh size in x at time $t = t_n$.

We apply the cubic spline technique derived in Section 2 to find the solution at next time step. By the nature of the solution, as t increases, the center of the shock starts shifting. If x_M is the center of the shock at last time we have rezoned and $x_{M'}$ is the center of the shock at the latest time, then $|M - M'|$ will increase. When $|M - M'| > 15$ or γ has changed by more than 10%, we again rezone the mesh. After rezoning we use cubic spline interpolation procedure to find u on the new mesh system. The method is repeated at later times.

5. Results and Discussion

The proposed variable mesh cubic spline technique is employed to study the propagation of the N-wave for the following three cases:

Case I: $\delta = 10^{-2}$, $\alpha = 0.2$, $\beta = 5.0$

Case II: $\delta = 10^{-3}$, $\alpha = 0.4$, $\beta = 2.5$.

Case III: $\delta = 10^{-4}$, $\alpha = 0.1$, $\beta = 5.0$.

For $\delta = 10^{-2}$ and $\delta = 10^{-3}$ we took

$$t_0 = \exp(1/4\delta) \tag{25}$$

so that at initial time $t = 1$, the shock is at $x = 0.5$. For $\delta = 10^{-4}$, due to computational difficulty, we could not use (25) for the constant t_0; hence we have taken $t_0 = 10^{300}$ so that at $t = 1$, the shock appears at $x = 0.2625$.

Figures 1–3 represent the propagation of the shocks for $\delta = 10^{-2}$, $\delta = 10^{-3}$, and $\delta = 10^{-4}$, respectively. For $\delta = 10^{-2}$, the shock is observed to be smooth at $t = 1$ and becomes smoother as time progresses. For $\delta = 10^{-3}$, the initial shock is found to be sharp and it becomes slightly smooth as time increases. While for $\delta = 10^{-4}$, the initial shock is very steep and remains so with a little smoothness for higher values of t. As time passes, the center of the shock in each case moves toward the right and the shock thickness increases. For $\delta = 10^{-4}$, the increment in the shock thickness, as time progresses, is smaller compared to the case of $\delta = 10^{-3}$ and $\delta = 10^{-2}$. The numerical values of the center of the shock and the shock thickness are represented in Table I (for $\delta = 10^{-3}$) and Table II (for $\delta = 10^{-4}$). The exact and the numerical values of the lobe Reynolds number R and maximum u are included in these tables for various values of time t. A close look at these tables reveals that the numerical results obtained by the proposed variable mesh cubic spline technique gives satisfactory

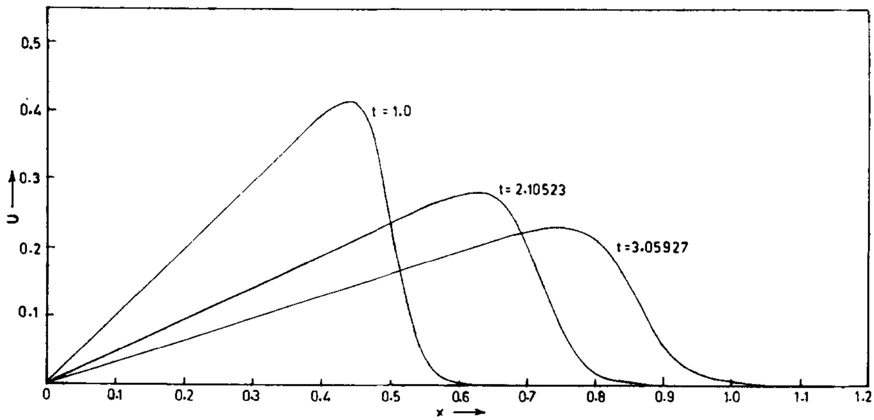

Fig. 1. Solution of Burgers' equation, $\delta = 10^{-2}$, $\alpha = 0.2$, $\beta = 5.0$.

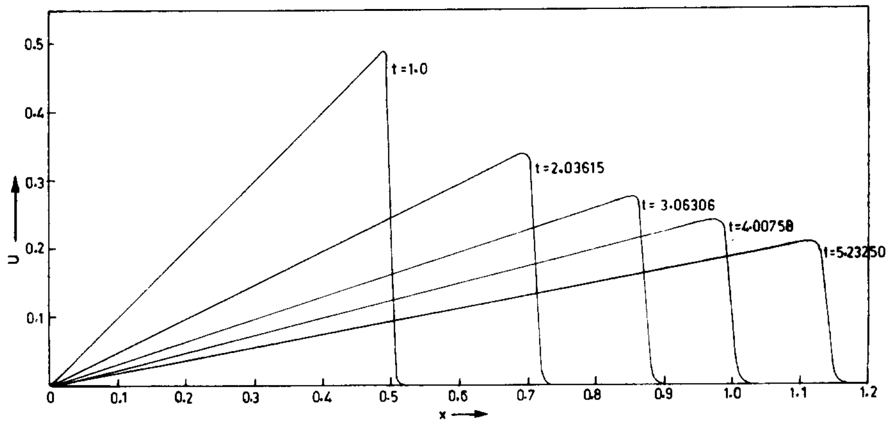

FIG. 2. Solution of Burgers' equation, $\delta = 10^{-3}$, $\alpha = 0.4$, $\beta = 2.5$.

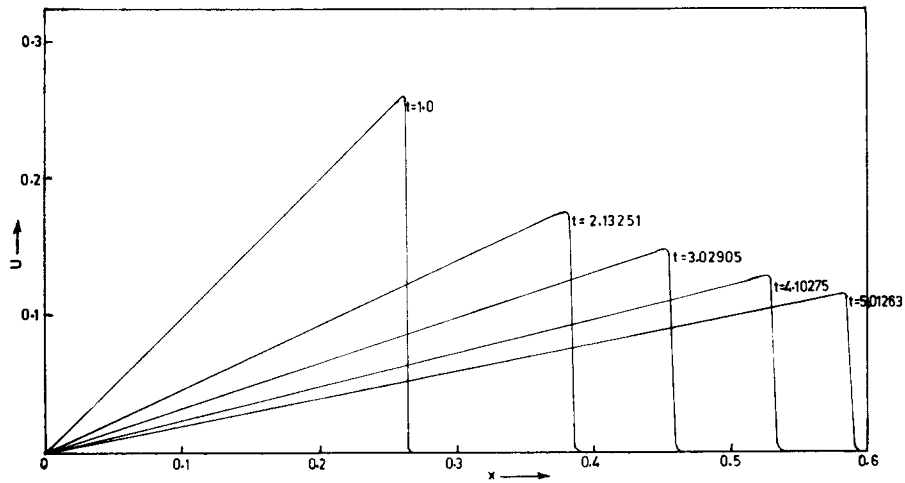

FIG. 3. Solution of Burgers' equation, $\delta = 10^{-4}$, $\alpha = 0.1$, $\beta = 5.00$.

results to an accuracy of $O(10^{-4})$. Total number of mesh points needed for calculations are also shown in the tables.

The proposed variable mesh cubic spline technique can be extended to solve problems with singularities. For example, consider (1) along with the initial condition [13]

$$u(x, 0) = 1, \qquad 0 \leqslant \pi \leqslant 0.4,$$
$$= 0, \qquad 0.4 < x \leqslant 1.0. \tag{26}$$

TABLE I

Comparison of Exact and Numerical Results for
$\delta = 10^{-3}$, $\alpha = 0.4$, $\beta = 2.5$

t	R exact	R numerical	Max. u (computed)	Max. error in u	Number of mesh points	Center of the shock	Shock thickness
1.0	125.0	124.99999	0.48688	–	409	0.5	0.0182
2.03615	124.64446	124.64402	0.34055	0.00015	390	0.7125	0.0350
3.06306	124.44029	124.43906	0.27728	0.00026	382	0.08725	0.0522
4.00758	124.30590	124.30557	0.24228	0.00032	378	0.9975	0.0677
5.23250	124.17255	124.17205	0.21170	0.00041	376	1.1400	0.0871

TABLE II

Comparison of Exact and Numerical Results for
$\delta = 10^{-4}$, $\alpha = 0.1$, $\beta = 5.0$

t	R exact	R numerical	Max. u (computed)	Max. error in u	Number of mesh points	Center of the shock	Shock thickness
1.0	345.38776	345.38787	0.25995	–	292	0.2625	0.0193
2.13251	345.00911	345.00930	0.17810	0.00019	293	0.3825	0.0304
3.02905	344.83363	344.83388	0.14954	0.00031	295	0.4575	0.0352
4.02905	344.68193	344.68227	0.12863	0.00043	296	0.5325	0.0431
5.01263	344.58178	344.58210	0.11656	0.00060	302	0.5875	0.0479

The analytical solution of this problem given by Lighthill [4] represents a shock wave progressing along the x axis. Davis [13] found the solution of (1) for the initial data (26) by applying Galerkin's method. The variable mesh cubic spline technique can well be applied for the numerical solution of this problem.

Acknowledgments

The authors gratefully acknowledge the financial support extended by the Department of Atomic Energy, Government of India, for carrying out the work on the computers CDC3600 and DEC10 situated at Tata Institute of Fundamental Research, Bombay.

References

1. J. M. Burgers, A mathematical model illustrating the theory of turbulence, *in* "Advances in Applied Mechanics" (R. V. Mises and T. V. Karman, eds.), Vol. 1, p. 171, Academic Press, New York, 1948.

2. E. HOPF, *Comm. Pure Appl. Math.* **3** (1950), 201.
3. J. D. COLE, *Quart. Appl. Math.* **9** (1951), 225.
4. M. J. LICHTHILL, Viscosity effects in sound waves of finite amplitude, *in* "Surveys in Mechanics" (G. K. Batchlor and R. M. Davis, Eds.), p. 250, Academic Press, New York and London, 1956.
5. P. L. SACHDEV AND SEEBASS, *J. Fluid Mech.* **58** (1973), 192.
6. J. DOUGLAS, JR. AND B. F. JONES, JR., *J. Soc. Indust. Appl. Math.* **11** (1963), 195.
7. P. C. JAIN AND D. N. HOLLA, *Int. J. Non-Linear Mech.* **13** (1978), 213.
8. P. C. JAIN AND B. L. LOHAR, *Comp. Math. Appl.* **5** (1979), 179.
9. T. H. CHONG, *SIAM. J. Numer. Anal.* **15** (1978), 835.
10. J. H. AHLBERG, E. N. NILSON, AND J. L. WALSH, "The Theory of Splines and Their Applications," Academic Press, New York, 1967.
11. R. M. BEAM AND R. F. WARMING, *J. Comput. Phys.* **22** (1976), 87.
12. L. LAPIDUS, "Digital Computation for Chemical Engineers," McGraw–Hill, New York, 1962.
13. A. M. DAVIES, *Comp. Meth. Appl. Mech. Eng.* **14** (1978), 305.